Multifractality of probability measure on energy-spectrum supports

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We have studied quadric type energy (frequency) recursion relations resulting from fractal lattices. We found that the allowed energy intervals in successive levels form two similar two-scale Cantor sets. If we imagine the iterating procedure as a dynamical process, the iterating results in different levels generate a "time" sequence, and we can introduce an equal probability measure p_i on a Cantor set and construct, following Helsey *et al.* [Phys. Rev. A **33**, 1141 (1986)], a partition function $\Gamma(q,\tau)=\sum_i p_i^q/l_i^\tau$; finally, we obtain D_q-q and $f(\alpha)$-α curves. A number of exactly soluable examples are investigated.

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I. INTRODUCTION

The concept of multifractality, which describes the distribution of singularitries of fractal measures, has been shown to be important and useful. Multifractal features are exhibited in many physical phenomena, such as in diffusion-limited aggregation the growth probability of perimeter sites of an aggregation cluster [1]; in percolation the distribution of voltages across the different elements in a random-resister network [2]; in dynamical systems the probability of visiting a given region of a strange attractor, and so on. The formulism of multifractal measures, which determine the distribution function $f(\alpha)$ of singularities of strength α for a given measure, has been presented [3,4].

We would like to see if there is similar nature for probability measure supported by the energy spectrum of an electron on a fractal. Domany *et al.* [5] and Rammal [6,7] have independently investigated the solution of the Schrödinger equation with a tight-binding Hamiltonian; they found that for a linear chain, Sierpinski gasket (SG) embedded in two- and three-dimensional Euclidean spaces and Berker lattices (hierarchical lattice) there is a common feature: The energy spectrum of an electron is divided into two parts, one of which obeys the recursion relation and another one includes only some special isolated values of energy, for which the recursion relation is inapplicable.

In this paper we are particularly interested in the first part of the energy spectrum mentioned above. Our research shows that the allowed energy intervals in successive levels approximately form two two-scale Cantor sets. If we imagine the iterating procedure as a dynamical process, (the iterating results in different levels generate a "time" sequence), we can assign an equal probability measure on each Cantor set and then obtain an $f(\alpha)-\alpha$ curve according to the formula given by Refs. [3,4]. We hope that the results may be useful for gaining some understanding of a kind of tight-binding type problem on fractal lattices.

II. THEORY AND GENERAL RESULTS

Following Refs. [5,6], we consider a set of nearest-neighbor hopping Hamiltonians of the form

$$H=u\sum_i |i\rangle\langle i|-t\sum_{\langle i,j\rangle}(|i\rangle\langle j|+|j\rangle\langle i|) \tag{2.1}$$

and work on some regular fractal lattices, where $|i\rangle$ is a local site function and u and t are on-site energy and hopping parameters, respectively. Using decimation transformation in a Hilbert space spanned by the local site functions $|i\rangle$, one produces a quadratic recursion relation for the eigenvalue of the Schrödinger equation like

$$\varepsilon_{n-1}=-\varepsilon_n^2+b_1\varepsilon_n+b_0 \ . \tag{2.2}$$

Here the subscript n denotes the nth stage of the construction of fractal lattices, and b_1 and b_0 are constants. The fixed points of Eq. (2.2) may be written as ε_-^* and ε_+^* (let $\varepsilon_+^*>\varepsilon_-^*$); they are

$$\varepsilon_\pm^*=\frac{(b_1-1)\pm[(b_1-1)^2+4b_0]^{1/2}}{2} \ . \tag{2.3}$$

In the examined examples, e.g., linear chain, two- and three-dimensional SG and Berker lattices, the recursion relation (2.2) has no stable fixed point or cycle. This means no interval of ε maps onto itself under repeated applications of the recursion relation. In more detail, ε will go toward negative infinity for the region I_1, i.e., $\varepsilon<\varepsilon_-^*$ and $\varepsilon>\varepsilon_+^*+1$, and thus the allowed energy interval will be $[\varepsilon_{11},\varepsilon_{12}]=[\varepsilon_-^*,\varepsilon_+^*+1]$ for the level-1 construction. Further, there is a smaller interval $[\varepsilon_{21},\varepsilon_{22}]$ within $[\varepsilon_{11},\varepsilon_{12}]$ which maps under one time application of recursion relation (2.2) onto a subset of I_1 and then the allowed regions will be $[\varepsilon_{11},\varepsilon_{21}]$ and $[\varepsilon_{22},\varepsilon_{12}]$ (see Fig. 1) for the level-2 construction, the ε_{21} and ε_{22} are determined by substituting the edge value ε_{12} into the left-hand side of Eq. (2.2),

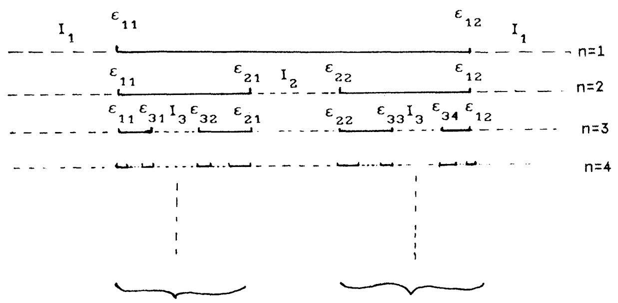

FIG. 1. Allowed intervals (solid line) and gaps (dashed line) of the spectrum obtained from the recursion relation (2.2) and two two-scale Cantor sets are shown. (a) For two-dimensional SG, $\varepsilon_{11}=-4$, $\varepsilon_{12}=1$; $\varepsilon_{21}=(-3-\sqrt{5})/2$, $\varepsilon_{22}=(-3+\sqrt{5})/2$; $\varepsilon_{31}=-3.706$, $\varepsilon_{32}=-3.1225$, $\varepsilon_{33}=0.1225$, $\varepsilon_{34}=0.706$. (b) For three-dimensional SG, $\varepsilon_{11}=-6$, $\varepsilon_{12}=0$; $\varepsilon_{21}=-4.732$, $\varepsilon_{22}=-1.268$; $\varepsilon_{31}=-5.78$, $\varepsilon_{32}=-5.066$, $\varepsilon_{33}=-0.934$, $\varepsilon_{34}=-0.2195$.

$$\varepsilon_{\substack{2\,1(-)\\2\,2(+)}}=\frac{b_1\pm(b_1^2-2\{(b_1+1)+[(b_1-1)^2+4b_0]^{1/2}\})^{1/2}}{2}\ . \tag{2.4}$$

Continuuing in this manner, we generate all different gaps in different stages of construction. In the limit $n\to\infty$, the allowed intervals shrink into a set of measure zero eventually.

We now focus on the allowed regions for different levels in the successive iterating process. We call these regions "allowed" energy intervals because in the given level the energy values within the region will map onto the allowed values in the previous level. In Fig. 1 we give a sketch which shows that the allowed energy intervals form two two-scale Cantor sets.

III. EXACTLY SOLUBLE EIGENVALUE PROBLEM ON FRACTAL LATTICES

We now proceed to apply the general results to some exactly soluble systems. Our emphasis is put on the observation of the two-scale Cantor set and the analysis of multifractality.

A. Two-dimensional Sierpinski gasket

A Sierpinski gasket can be constructed as follows: Start with a generator, shown in Fig. 2, then insert sites into the generator so that the scale changes to 2. Continuing in the same way, we obtain the nth level SG ($n=1, 2, \ldots$). The boundary conditions identify the corners of two triangles on the largest scale.

Solving the eigenvalue equation associated with the tight-binding Hamiltonian (2.1) and employing the dimensionless energy parameter finally produces the recursion relation for the eigenvalues [5]

$$\varepsilon_{n-1}=-\varepsilon_n^2-3\varepsilon_n\ , \tag{3.1}$$

where ε_i represents the eigenvalue of the level-i system. Comparing (3.1) with (2.2) we found $b_1=-3$, $b_0=0$, and

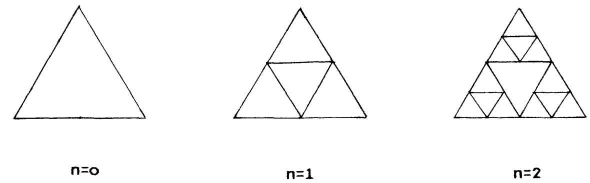

FIG. 2. Two-dimensional SG; $n=1$ stage of construction corresponds to the generator.

thus, according to expression (2.3), the fixed points $\varepsilon_-^*=-4$, $\varepsilon_+^*=0$. Following the results in Sec. II, we present a sketch (see Fig. 1) of the forbidden energy regions and allowed intervals in each stage of construction, which shows the allowed energy regions $[-4,1]$ in the level-1 stage, $[-4,(-3-\sqrt{5})/2]$ and $[(-3+\sqrt{5})/2,1]$ in the level-2 stage, and so forth. Numerical calculation tells us that there are two approximate common length rescaling factors for the size of the allowed energy intervals between successive neighboring levels and with which the allowed energy intervals form two similar two-scale Cantor sets. In Fig. 1 we quote a number of data up to the level-3 system, and choose the common (average) length rescaling factors between successive neighboring levels to be 0.353 and 0.225, respectively.

Let us just take one of the Cantor sets into account and suppose a measure can be generated by the following process. Start with the allowed energy interval $[(-3+\sqrt{5})/2,1]$ of the level-2 system, which has measure 1 and size 1.382, which is the length of the interval. Divide the region in terms of the application of Eq. (3.1) into two allowed pieces with size 0.504 and 0.294, an equal probability measure of 0.5, and one gap. Repeating the process, a measure possessing a recursive structure is obtained and then one can find $f(\alpha)$, the distribution function of singularities with strength α of the measure, or equivalently the D_q, a generalized dimension, in which D_0 is just the fractal dimension of the support of the measure while D_1 is the information dimension and D_2 is the correlation dimension [3,4].

For performing the mentioned points, we can construct a partition function for the measure

$$\Gamma(q,\tau)=\left[\frac{p_1^q}{l_1^\tau}+\frac{p_2^q}{l_2^\tau}\right]^m\ ,\quad m=n-2\ ,n>2\ , \tag{3.2}$$

where $l_1=0.353$, $l_2=0.225$, and $p_1=p_2=0.5$. Let $\Gamma(q,\tau)=1$; we obtain $\tau=\tau(q)$. Follow the standard procedure [3], the plot of D_q vs q and the plot of $f(\alpha)$ vs α for the set are shown in Figs. 3 and 4.

B. Three-dimensional SG

Take a basic object, e.g., a tetrahedon, insert six sites into each edge, remove the central small tetrahedon, and then form a generator of three-dimensional SG (see Fig. 5). Continuing the iterating process produces a level-n SG. We suppose that the corners are shared by two tetrahedons on the largest scale. In the similar study used before one gets the recursion relation of the dimen-

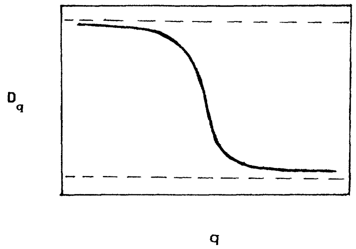

FIG. 3. A sketch of D_q vs q for the examined examples in the text; the scale is not the same for different examples.

sionless energy eigenvalues as follows:

$$\varepsilon_{n-1} = -(\varepsilon_n^2 + 6\varepsilon_n + 6) \; ; \tag{3.3}$$

this is also quadratic and its fixed points are $\varepsilon_-^* = -6$ and $\varepsilon_+^* = -1$. Similar to the two-dimensional SG, the allowed energy interval of the level-1 system which corresponds to the generator of three-dimensional SG is $[\varepsilon_{11}, \varepsilon_{12}] = [-6, 0]$. For the next level, the allowed values will be $[-6, -3-\sqrt{3}]$ and $[-3+\sqrt{3}, 0]$. Repeating the iteration with the inverse of the recursion relation (3.3), we finally obtain the forbidden gaps and the allowed energy regions for all different levels. Once again we find two similar two-scale Cantor sets with common (average) length rescale factors of 0.176 and 0.26, respectively.

As we have done before, a measure with hiearchical structure may be introduced and a partition function similar to (3.2) can be defined. Substituting $p_1 = p_2 = 0.5$, $l_1 = 0.176$, and $l_2 = 0.26$ into expression (3.2), we finally obtain the curves $f(\alpha)$ and D_q.

In summary, for the Schrödinger eigenvalue equation associated with the tight-binding type Hamiltonian on a SG the energy spectrum includes two two-scale Cantor sets with the same rescaling factors. By defining a measure with hierarchical structure and employing the standard procedure [3,4] we can determine the generalized dimension D_q and the distribution function $f(\alpha)$.

C. Berker lattice

The Berker lattice is a hierarchical lattice and is generated in the manner indicated in Fig. 6. To insert a new level of sites, replace each bond of the lattice by two bonds with a site centered on each.

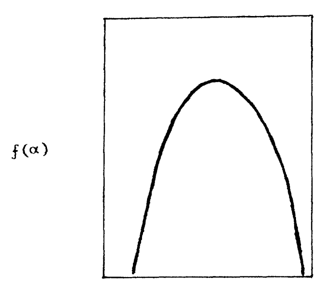

FIG. 4. A plot of $f(\alpha)$ vs α for the examined examples in the text; the scale is not the same for different examples.

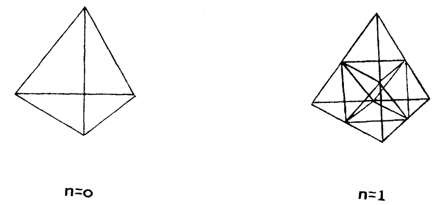

FIG. 5. Three-dimensional SG; $n=1$ stage of construction corresponds to the generator.

When we find the solution of the Schrödinger eigenvalue equation with the tight-binding Hamiltonian, we must note that the coordination number is different for sites belonging to different generations, so in order to derive a recursion formula of the eigenvalues, Domany *et al.* [5] assume that the hopping parameters depend on the location of the respective site on the lattice and result in a simple recursion relation

$$\varepsilon_{n-1} = -\varepsilon_n^2 + 2 \; , \tag{3.4}$$

which corresponds to $b_1 = 0$, $b_0 = 2$ in expression (2.2). The fixed points are $\varepsilon_+^* = 1$, $\varepsilon_-^* = -2$, and therefore the allowed energy interval is $[\varepsilon_{11}, \varepsilon_{12}] = [-2, 2]$ for the level-1 stage.

In this example, a novel feature appears: Let us take the values within $[-2,2]$ and substitute them into the left-hand side of (3.4); we find that two new allowed energy intervals merge and thus new forbidden regions never appear. Therefore the allowed energy interval for the level-1 system will apply to any level and the Cantor set structure will not appear.

We would like to consider a general problem, i.e., $b_1 = 0$ and $b_0 > 0$. It is easy to find that the allowed eigenvalue intervals are always symmetric for any stage of the structure and there are two two-scale Cantor sets except for $b_0 = 2$. In fact, when $b_0 = 2 + \theta$, where θ is orbitrary values (even if infinitesimal), the forbidden gap will generate and enlarge with the increase of θ.

D. Lattice vibrations on SG

We now apply the above theory to a dynamical problem—elastic vibrations on two-dimensional SG. Since the equations of motion for elastic vibrations (one equation for each site) are similar to the Schrödinger eigenvalue equations associated with the Hamiltonian (2.1),

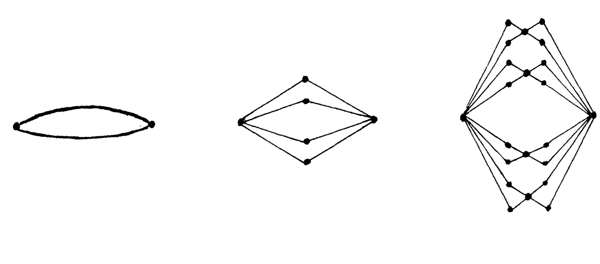

FIG. 6. The construction process of Berker lattice; the level-1 lattice is the generator.

a similar decimation procedure leads to the recursion equation for the eigenvalue of frequency

$$\varepsilon_{n-1} = -\varepsilon_n^2 + 5\varepsilon_n \ , \quad \text{for } d=2 \ . \tag{3.5}$$

Using the formula (2.3), we obtain the fixed points $\varepsilon_-^* = 0$ and $\varepsilon_+^* = 4$, and thus the allowed interval is $[\varepsilon_{11}, \varepsilon_{12}] = [0, 5]$ for the level-1 SG. Since the negative values of frequency are not allowed, the allowed eigenvalues are restricted. Repeating the application of (3.5), we generate the allowed and forbidden frequencies for all different level-n stages which form two two-scale Cantor sets with a suitable choice of common length rescaling factors 0.224 566 7 and 0.352 101 3. We also obtain the curves D_q and $f(\alpha)$.

IV. CONCLUSIONS

We have investigated a kind of recursion relations (2.2) and its inverse with the restrict: The iterating results do not approach $-\infty$ for the electron spectrum and do not take negative values for elastic vibration. The general character of the energy (frequency) spectrum is the existence of two two-scale Cantor sets in which the common rescaling factors of each approximately equal some average values. The choice of the average values is such that the relative error is as small as possible. We have also introduced a probability measure possessing an exact recursive structure and found the curves D_q and $f(\alpha)$ for a given measure.

Mathematically, the above recursion relation represents maps in a real number region, and therefore the results also reflect the general feature of the maps. We believe that our finding will promote the understanding of a kind of physical problem, not only a tight-binding type one on fractal lattice but also other dynamical problems, perhaps.

ACKNOWLEDGMENTS

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